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# Local quadrature formulas on the sphere

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## Abstract

Let  $q \geq 1$  be an integer,  $\mathbb{S}^q$  be the unit sphere embedded in  $\mathbb{R}^{q+1}$ , and  $\mu_q$  be the volume element of  $\mathbb{S}^q$ . For  $\mathbf{x}_0 \in \mathbb{S}^q$ , and  $\alpha \in (0, \pi)$ , let  $\mathbb{S}_\alpha^q(\mathbf{x}_0)$  denote the cap  $\{\xi \in \mathbb{S}^q : \mathbf{x}_0 \cdot \xi \geq \cos \alpha\}$ . We prove that for any integer  $m \geq 1$ , there exists a positive constant  $c = c(q, m)$ , independent of  $\alpha$ , with the following property. Given an arbitrary set  $\mathcal{C}$  of points in  $\mathbb{S}_\alpha^q(\mathbf{x}_0)$ , satisfying the mesh norm condition  $\max_{\xi \in \mathbb{S}_\alpha^q(\mathbf{x}_0)} \min_{\zeta \in \mathcal{C}} \text{dist}(\xi, \zeta) \leq c\alpha$ , there exist nonnegative weights  $w_\xi$ ,  $\xi \in \mathcal{C}$ , such that

$$\int_{\mathbb{S}_\alpha^q(\mathbf{x}_0)} P(\xi) d\mu_q(\xi) = \sum_{\xi \in \mathcal{C}} w_\xi P(\xi)$$

for every spherical polynomial  $P$  of degree at most  $m$ . Similar quadrature formulas are also proved for spherical bands.

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## 1. Introduction

In many practical applications, one needs to evaluate integrals on the sphere embedded in a Euclidean space [4,5]. Such evaluations are necessary, for example, in using Galerkin methods to solve partial differential equations on the sphere. Often, these integrals cannot be evaluated analytically, and quadrature formulas must be used. Many quadrature formulas are developed in the literature.

In the case of “equal-angle” sites, Driscoll and Healy [2] and Potts et al. [13] obtained quadrature formulas on  $\mathbb{S}^q$  that are exact for high degree polynomials, and with explicitly calculated weights. In [12], Petrushev has described quadrature

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formulas that use specific sites, which are not “equal-angle”, that collect around the poles of  $S^q$ . More recently, Brown et al. [1] have obtained similar formulas based on specific sites, where the weights are explicitly described.

In many applications, in particular, those connected with neural networks, one has no control on the choice of sites, and must therefore, deal with *scattered data*. Jetter et al. [6] obtained quadrature formulas for scattered sites; these formulas used weights that were real, but possibly negative. In [9], we obtained quadrature formulas valid for high degree polynomials, based on scattered sites, and having nonnegative weights that can be computed using quadratic or linear programming techniques. These were of crucial importance in our work on approximation on the sphere using zonal function networks [8], and analysis of data using frames consisting of polynomials [11] and zonal function networks [10].

To the best of our knowledge, there are no known quadrature formulas in the case when the data are available on a part of the sphere, and the integrals are required on this part of the sphere. Such formulas are expected to be useful in local approximation on the sphere as well as analyzing local data on the sphere. In this paper, we prove the existence of quadrature formulas with nonnegative weights that are exact for evaluating integrals of polynomials of a fixed degree on a spherical cap.

In the next section, we discuss our main theorem. In Section 3, we review certain preliminary results which are used in the proof of the main theorem. The proof is given in Section 4.

## 2. Main theorem

In the remainder of this paper,  $q \geq 1$  and  $m \geq 1$  are fixed integers. We adopt the following convention regarding constants. The symbols  $c, c_1, \dots$  will denote positive constants depending only on  $q$  and  $m$ , and other explicitly mentioned quantities. The dependence on  $m$  will be polynomial, and the values of these constants may be different at different occurrences, even within a single formula. The fact that  $A \leq c_1 B \leq c_2 A$  will be denoted by  $A \sim B$ .

We will need certain notions regarding the data on three spheres, embedded in  $\mathbb{R}^2$ ,  $\mathbb{R}^q$ ,  $\mathbb{R}^{q+1}$ . Since  $q$  is a fixed number, we will describe these notions for a general  $d$ -dimensional sphere. The symbol  $\mathbb{S}^d$  denotes the unit sphere embedded in  $\mathbb{R}^{d+1}$  and  $\mu_d$  denotes its volume (surface area) measure. The class of all spherical polynomials on  $\mathbb{S}^d$  of total degree at most  $n$  is denoted by  $\Pi_n^d$ . If  $A \subseteq \mathbb{S}^d$ , and  $\mathcal{C}$  is a set of distinct points in  $A$ , we define the mesh norm of  $\mathcal{C}$  relative to  $A$  by

$$\delta_d(\mathcal{C}, A) := \max_{\mathbf{x} \in A} \min_{\xi \in \mathcal{C}} \text{dist}_d(\mathbf{x}, \xi), \quad (2.1)$$

where  $\text{dist}_d(\mathbf{x}, \xi)$  is the geodesic distance between  $\mathbf{x}$  and  $\xi$  with respect to  $\mathbb{S}^d$ . If  $\mathbf{x}_0 \in \mathbb{S}^d$ , the spherical cap centered at  $\mathbf{x}_0$  and radius  $\alpha \in [0, \pi]$  is defined by

$$\mathbb{S}_\alpha^d(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{S}^d : \mathbf{x} \cdot \mathbf{x}_0 \geq \cos \alpha\}, \quad (2.2)$$

where  $\cdot$  is the usual inner product in  $\mathbb{R}^d$ . (To keep the notation simple, we will not mention the dependence of this inner product on  $d$ ; it will always be clear from the context.)

Our main theorem is the following:

**Theorem 2.1.** *Let  $q \geq 1$ ,  $m \geq 1$  be integers,  $\alpha \in (0, \pi)$ ,  $\mathbf{x}_0 \in \mathbb{S}^q$  and  $\mathcal{C}$  be a set of distinct points in  $\mathbb{S}_\alpha^q(\mathbf{x}_0)$ . There exists a positive constant  $c = c(q, m)$ , independent of  $\mathbf{x}_0$  or  $\alpha$ , with the following property. If  $\delta_q(\mathcal{C}, \mathbb{S}_\alpha^q(\mathbf{x}_0)) \leq c\alpha$ , then there exist nonnegative weights  $w_\xi$ ,  $\xi \in \mathcal{C}$ , such that*

$$\sum_{\xi \in \mathbb{S}_\alpha^q(\mathbf{x}_0)} w_\xi P(\xi) = \int_{\mathbb{S}_\alpha^q(\mathbf{x}_0)} P(\mathbf{x}) d\mu_q(\mathbf{x}), \quad P \in \Pi_m^q. \quad (2.3)$$

Moreover,

$$|\{\xi \in \mathcal{C} : w_\xi \neq 0\}| \leq c_1. \quad (2.4)$$

In particular,

$$\max_{\xi \in \mathcal{C}} w_\xi \sim \sum_{\xi \in \mathcal{C}} w_\xi \sim \alpha^q. \quad (2.5)$$

The central ideas behind our proof of Theorem 2.1 are essentially the same as those in [9]. We prove a Marcinkiewicz–Zygmund inequality estimating the  $L^1$  norm of polynomials on the cap in terms of a discrete  $\ell^1$  norm of the values of the polynomials at the sampling sites. A very general theorem regarding norming sets then yields the quadrature formulas and other facts stated in Theorem 2.1. The technical details are quite different from those in [9], partly because there is no known reproducing kernel for polynomials on caps.

Given points  $\{\xi_1, \dots, \xi_N\}$  on a cap  $\mathbb{S}_\alpha^q(\mathbf{x}_0)$ , one way to evaluate the corresponding weights  $\{w_1, \dots, w_N\}$  is the following. Starting with  $L = 1$ , we increase the value of  $L$  one by one until the following quadratic programming problem no longer has a feasible solution:

$$\text{minimize } \sum_{j=1}^N w_j^2$$

subject to the constraints

$$\sum_{j=1}^N w_j P_m(\xi_j) = \int_{\mathbb{S}_\alpha^q(\mathbf{x}_0)} P_m(\mathbf{x}) d\mu_q(\mathbf{x}), \quad m = 0, \dots, \text{dimension}(\Pi_L^q),$$

$$w_j \geq 0, \quad j = 1, \dots, N,$$

where  $\{P_m\}$  is some basis for  $\Pi_L^q$ .

A natural choice of a basis is described as follows. We recall that for integer  $\ell \geq 0$ , the restriction to  $\mathbb{S}^q$  of a homogeneous harmonic polynomial of degree  $\ell$  is called a

spherical harmonic of degree  $\ell$ . The class of all spherical harmonics of degree  $\ell$  will be denoted by  $\mathbf{H}_\ell^q$ . The dimension of  $\mathbf{H}_\ell^q$  is given by

$$d_\ell^q := \dim \mathbf{H}_\ell^q = \begin{cases} \frac{2\ell+q-1}{\ell+q-1} \binom{\ell+q-1}{\ell} & \text{if } \ell \geq 1, \\ 1 & \text{if } \ell = 0. \end{cases} \quad (2.6)$$

If we choose an orthonormal basis  $\{Y_{\ell,k} : k = 1, \dots, d_\ell^q\}$  for each  $\mathbf{H}_\ell^q$ , then the set  $\{Y_{\ell,k} : \ell = 0, 1, \dots, L, k = 1, \dots, d_\ell^q\}$  is an orthonormal basis for  $\Pi_L^q$ . In the important case when  $q = 2$ , it is customary to let  $k = -\ell, \dots, \ell$  instead of  $1, \dots, 2\ell + 1$ . We adopt the spherical coordinates:

$$x = \sin \theta \cos \varphi, \quad y = \sin \theta \sin \varphi, \quad z = \cos \theta.$$

Following [3, Section 2.5], the spherical harmonic  $Y_{\ell,k}$  is then defined by (2.7), (2.9) as a function of  $\theta, \varphi$ : For  $\ell = 0, 1, 2, \dots$  and  $k = 0, 1, \dots, \ell$ , let

$$Y_{\ell,k}(\theta, \varphi) := (-1)^k \sqrt{\frac{(2\ell+1)(\ell-k)!}{4\pi(\ell+k)!}} P_\ell^k(\cos \theta) e^{ik\varphi}, \quad (2.7)$$

where  $P_\ell^k(\cos \theta) := \sin^k \theta P_\ell^{[k]}(\cos \theta)$  is the associated Legendre function and  $P_\ell^{[k]}$  is  $k$ th derivative of the usual Legendre polynomial  $P_\ell$ , defined recursively by

$$P_\ell(x) = (2 - 1/\ell)xP_{\ell-1}(x) - (1 - 1/\ell)P_{\ell-2}(x), \quad \ell = 2, 3, \dots, \quad (2.8)$$

$$P_0(x) = 1, \quad P_1(x) = x.$$

For negative  $k$ , we use the identity [3, Eq. (2.5.6)]

$$Y_{\ell,k}(\theta, \varphi) = (-1)^k \overline{Y_{\ell,-k}(\theta, \varphi)}. \quad (2.9)$$

For the cap  $\mathbb{S}_\alpha^q((0, 0, 1))$ , we have

$$\begin{aligned} & \int_{\mathbb{S}_\alpha^q((0,0,1))} Y_{\ell,k}(\mathbf{x}) d\mu_2(\mathbf{x}) \\ &= 2\pi Y_{\ell,k}(0, 0) \begin{cases} \frac{1 - \cos^2 \alpha}{2\ell} P_{\ell-1}^{(1,1)}(\cos \alpha) & \text{if } \ell \geq 1, \\ 1 - \cos \alpha & \text{if } \ell = 0, \end{cases} \end{aligned} \quad (2.10)$$

where  $P_m^{(1,1)}$  are the Jacobi polynomials defined in [14, Eq. (4.5.1), p. 71].

In our numerical experiments, we found it convenient to normalize the integrals above by dividing by the volume of the cap. We used the routine `quadprog` in the optimization toolbox of Matlab 5.3 to solve the optimization problem above. We ran a slight variation of the above algorithm 100 times each for different values of  $\alpha$ , generating a different set of 36 random points each time. Table 1 summarizes the results. The columns correspond to different values of  $\alpha$ , the rows represent the frequency with which quadrature formulas exact for the given degree were obtained. The total time taken for this experiment, as measured by the `tic` and `toc` commands of Matlab 5.3 was 157.0800 s.

Since we are obtaining quadrature formulas for the average integral over the cap, it is not surprising that they could be obtained for higher degree polynomials more

Table 1  
Frequencies with which quadrature formulas exact for given degree were obtained in 100 experiments with different data sets of 36 points on different caps

Degree	$\alpha$				
	$\pi$	$\pi/2$	$\pi/128$	$\pi/256$	$\pi/1024$
1	0	0	1	4	0
2	27	42	39	42	27
3	72	58	57	52	50
4	1	0	3	2	23

Table 2  
Frequencies with which quadrature formulas exact for given degree were obtained in 100 experiments with different data sets consisting of given number of points on the cap  $\mathbb{S}_{\pi/128}^q((0, 0, 1))$

Degree	Points				
	20	40	60	80	100
1	48	1	0	0	0
2	52	36	5	0	0
3	0	58	43	12	1
4	0	5	49	36	8
5	0	0	3	46	49
6	0	0	0	6	35
7	0	0	0	0	7

frequently for the small cap corresponding to the last column. However, we find it surprising to notice the high consistency in the case of quadrature formulas on the whole sphere corresponding to the first column.

Next, we studied the effect of the number of points on the degree for the cap  $\mathbb{S}_{\pi/128}^q((0, 0, 1))$ . The results are tabulated in Table 2. The time requirement for this experiment was 795.87.

Finally, we took two sets of 40 points on two different caps which gave quadrature formulas of the maximum degree, and noted the distribution of the corresponding weights, shown in Fig. 1. The middle line gives the mean of the weights, and the upper and lower lines represent one standard deviation above and below the mean respectively.

3. Auxiliary results

In this section, we review some auxiliary results needed in our proofs. The results in Section 3.2 are probably not new, but except for Proposition 3.1, we find it easier to prove them rather than looking for an appropriate reference.

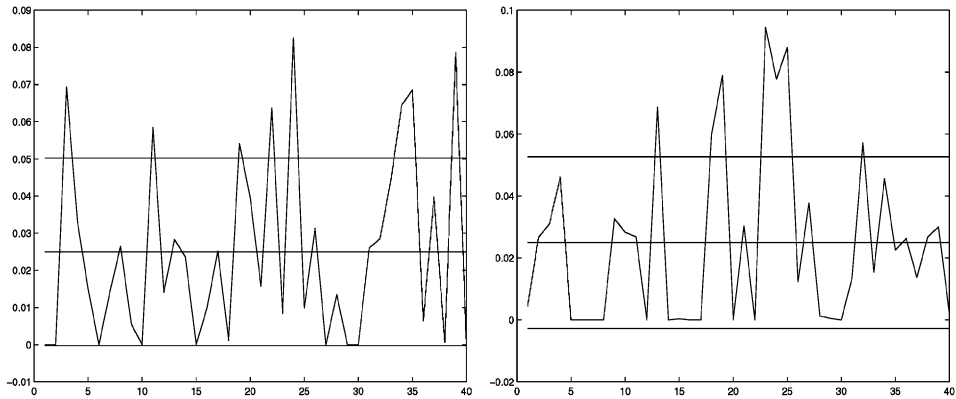


Fig. 1. Distribution of the 40 weights in a degree 4 quadrature formula for the cap of radius  $\pi/128$  (left) and a degree 5 quadrature formula for the cap of radius  $\pi/1024$  (right).

### 3.1. Norming sets

Let  $X$  be a finite dimensional normed linear space,  $X^*$  be its dual space,  $Z \subseteq X^*$  be a finite set of functionals. We say that  $Z$  is a *norming set* for  $X$  if the operator  $x \mapsto (y^*(x))_{y^* \in Z}$  is injective. A functional  $x^* \in X^*$  is said to be *positive with respect to*  $Z$  if  $x^*(x) \geq 0$  whenever  $y^*(x) \geq 0$  for all  $y^* \in Z$ .

In [9], we proved the following general theorem regarding the representation of functionals positive with respect to  $Z$ .

**Theorem 3.1.** *Let  $X$  be a finite dimensional normed linear space,  $X^*$  be its dual,  $Z \subseteq X^*$  be a finite, norming set for  $X$ , and  $x^* \in X^*$  be positive with respect to  $Z$ . Suppose further that there exists a  $x_0 \in X$  such that  $y^*(x_0) > 0$  for all  $y^* \in Z$ . Then there exist nonnegative numbers  $w_{y^*}$ ,  $y^* \in Z$ , such that*

$$x^*(x) = \sum_{y^* \in Z} w_{y^*} y^*(x), \quad x \in X. \quad (3.1)$$

We will use Theorem 3.1 with  $\Pi_m^q$  in place of  $X$ , take  $Z$  to be the set of point evaluation functionals at points of  $\mathcal{C}$ , and  $x^*$  to be the functional associating each polynomial with its integral over the cap. We observe that the polynomial identically equal to 1 serves in place of  $x_0$  in Theorem 3.1. The rest of the properties of  $Z$  will be obtained using Marcinkiewicz–Zygmund inequalities. The proof of these inequalities is therefore, the main objective of most of Section 4. As in [9], a critical ingredient in this proof is an analogue of the Bernstein inequality, which will be proved in the next subsection (Theorem 3.2).

### 3.2. Trigonometric polynomials on an arc

In this section, all norms will be taken on the circle with respect to the arclength. For any integer  $m \geq 0$ , we denote the class of all trigonometric polynomials of degree

at most  $m$  by  $\mathbb{H}_m$ . Let  $J \subseteq (-\pi, \pi]$  be an interval (arc on the circle), and  $|J|$  denote its length. We start with the following Videnski–Lubinsky inequalities (cf. [7, Theorem 1.1]):

**Proposition 3.1.** *Let  $1 \leq p \leq \infty$ ,  $J \subset (-\pi, \pi]$ ,  $m \geq 1$  an integer, and  $T \in \mathbb{H}_m$ . Then*

$$\|T'\|_{J,p} \leq c \frac{m^2}{|J|} \|T\|_{J,p}. \quad (3.2)$$

An important consequence of this inequality is the following Nikolskii-type inequality:

**Proposition 3.2.** *Let  $J \subset (-\pi, \pi]$ ,  $m \geq 1$  an integer, and  $T \in \mathbb{H}_m$ . Then*

$$\|T\|_{J,\infty} \leq \frac{cm^2}{|J|} \|T\|_{J,1}. \quad (3.3)$$

**Proof.** Let  $t_0 \in J$  and  $|T(t_0)| = \|T\|_{J,\infty}$ . Then for  $|t - t_0| \leq c|J|/m^2$ ,  $t \in J$ , we have using (3.2) :

$$|T(t) - T(t_0)| \leq |t - t_0| \|T'\|_{J,\infty} \leq \frac{cm^2}{|J|} |t - t_0| \|T\|_{J,\infty} \leq (1/2) |T(t_0)|.$$

Therefore  $|T(t)| \geq (1/2) \|T\|_{J,\infty}$  for all such  $t$ , and hence,

$$\int_J |T(t)| dt \geq \int_{t \in J, |t-t_0| \leq c|J|/m^2} |T(t)| dt \geq \frac{c|J|}{m^2} \|T\|_{J,\infty}. \quad \square$$

Next, we need an inequality to insert a sine factor in the integral.

**Proposition 3.3.** *Let  $J \subset (-\pi, \pi]$ ,  $m \geq 1$  an integer, and  $T \in \mathbb{H}_m$ . Then*

$$\|T\|_{J,1} \leq \frac{cm^2}{|J|} \|T \sin(\cdot)\|_{J,1}. \quad (3.4)$$

**Proof.** Let  $A \subseteq J$  be a measurable set, and  $B = J \setminus A$ . Then in view of (3.3)

$$\begin{aligned} \int_J |T(t)| dt &= \int_B |T(t)| dt + \int_A |T(t)| dt \\ &\leq \int_B |T(t)| dt + |A| \|T\|_{J,\infty} \\ &\leq \int_B |T(t)| dt + \frac{cm^2 |A|}{|J|} \|T\|_{J,1}. \end{aligned}$$

Consequently, we see that there exists a constant, to be denoted in this proof only, by  $\alpha$ , such that if  $|A| \leq \alpha|J|/m^2$  then

$$\int_J |T(t)| dt \leq 2 \int_{J \setminus A} |T(t)| dt. \quad (3.5)$$

In this proof only, let  $\delta := \alpha|J|/(4m^2)$ ,

$$A := J \cap ([-\delta, \delta] \cup [-\pi, -\pi + \delta] \cup [\pi - \delta, \pi]),$$

and  $B := J \setminus A$ . Clearly,  $|A| \leq \alpha|J|/m^2$ , and for  $t \in B$ ,  $|\sin t| \geq c|J|/m^2$ . Therefore, (3.5) implies that

$$\|T\|_{J,1} \leq 2 \int_B |T(t)| dt \leq \frac{cm^2}{|J|} \int_B |T(t) \sin t| dt. \quad \square$$

We are now in a position to prove an analogue of Proposition 3.1 with weight functions.

**Theorem 3.2.** *Let  $J \subset (-\pi, \pi]$ ,  $m \geq 1$ ,  $d \geq 1$  be integers, and  $T \in \mathbb{H}_m$ . Then*

$$\int_J |T'(t) \sin^d t| dt \leq \frac{c(d)m^2}{|J|} \int_J |T(t) \sin^d t| dt. \quad (3.6)$$

**Proof.** In this proof only, let  $R(t) = T(t) \sin^d t$ . Then

$$T'(t) \sin^d t = R'(t) - d \cos t \sin^{d-1} t T(t). \quad (3.7)$$

Since  $R \in \mathbb{H}_{m+d}$ , (3.2) implies that

$$\|R'\|_{J,1} \leq \frac{cm^2}{|J|} \|R\|_{J,1} = \frac{cm^2}{|J|} \int_J |T(t) \sin^d t| dt.$$

Also, (3.4) implies that

$$\int_J |\cos t \sin^{d-1} t T(t)| dt \leq \frac{cm^2}{|J|} \int_J |T(t) \sin^d t| dt.$$

Therefore, (3.7) leads to (3.6).  $\square$

### 3.3. Polynomials on the sphere

We will need the following concepts and results from [9], many of which will be used in the context of  $\mathbb{S}^1$ ,  $\mathbb{S}^{q-1}$ , and  $\mathbb{S}^q$ . In this subsection,  $d$  is an integer,  $1 \leq d \leq q$ . In this subsection only, for a finite set  $\mathcal{C} \subset \mathbb{S}^d$ , we will write  $\delta_{\mathcal{C}}$  to denote the mesh norm  $\delta_d(\mathcal{C}, \mathbb{S}^d)$ . Let  $\mathcal{C}_0$  be a set of distinct points on  $\mathbb{S}^d$ .



**Definition 3.1.** Let  $\mathcal{R}$  be a *finite* collection of closed, non-overlapping (i.e., having no common interior points) regions  $R \subset \mathbb{S}^d$  such that  $\bigcup_{R \in \mathcal{R}} R = \mathbb{S}^d$ . We will say that  $\mathcal{R}$  is  $\mathcal{C}_0$ -compatible if each  $R \in \mathcal{R}$  contains at least one point of  $\mathcal{C}_0$  in its  $(\mathbb{S}^d)$ -interior. The partition norm for  $\mathcal{R}$  is defined by

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \text{diam } R$$

If  $\mathcal{R}$  is a  $\mathcal{C}_0$ -compatible decomposition, we can choose one point  $\xi \in \mathcal{C}_0$  interior to each region. We can then use this point to label uniquely the region as  $R_\xi$ ; the set of such points will be denoted by  $\mathcal{C}$ . Of course,  $\mathcal{C} \subseteq \mathcal{C}_0$  and  $\mathcal{R} = \{R_\xi\}_{\xi \in \mathcal{C}}$ . Furthermore, no point in  $R_\xi$  can be farther from  $\xi$  than  $\text{diam } R_\xi \leq \|\mathcal{R}\|$ ; hence,  $\delta_{\mathcal{C}_0} < \|\mathcal{R}\|$ . Moreover, it is also easy to see that  $\mathcal{R}$  is  $\mathcal{C}$ -compatible, so  $\delta_{\mathcal{C}} < \|\mathcal{R}\|$ . Finally, removing points from  $\mathcal{C}_0$  only increases its mesh norm; hence, we have the following bounds:

$$\delta_{\mathcal{C}_0} \leq \delta_{\mathcal{C}} < \|\mathcal{R}\|. \quad (3.8)$$

**Proposition 3.4.** If  $\mathcal{C}_0$  and  $\delta_{\mathcal{C}_0}$  are as above, then there exists a  $\mathcal{C}_0$ -compatible decomposition  $\mathcal{R}$  for which each  $R \in \mathcal{R}$  is a spherical simplex. A reduced set  $\mathcal{C}$  can be found with each point  $\xi \in \mathcal{C}$  in the interior of  $R_\xi$ . In addition, the norm of  $\mathcal{R}$ , the mesh norm  $\delta_{\mathcal{C}_0}$ , and the common cardinality of  $\mathcal{R}$  and  $\mathcal{C}$  satisfy

$$2\delta_{\mathcal{C}_0} \leq \|\mathcal{R}\| \leq 8d\sqrt{2d(d+1)}\delta_{\mathcal{C}_0}, \quad (3.9)$$

$$|\mathcal{R}| = |\mathcal{C}| = 2^{d+1} \left\lceil \frac{1}{2d\sqrt{d+1}\delta_{\mathcal{C}_0}} \right\rceil^d, \quad (3.10)$$

$$\frac{2}{\sqrt{d(d+1)}} (|\mathcal{C}|/2)^{-\frac{1}{d}} \leq \|\mathcal{R}\| \leq 4\sqrt{2d} (|\mathcal{C}|/2)^{-\frac{1}{d}}. \quad (3.11)$$

Further, for any  $R \in \mathcal{R}$  and  $\mathbf{x}$  in the interior of  $R$ , any geodesic through  $\mathbf{x}$  intersects  $R$  in exactly two points.

Next, we describe certain Marcinkiewicz–Zygmund type inequalities. It is convenient to abbreviate the statements of the various theorems in this connection using the following definition.

**Definition 3.2.** Let  $\mathcal{C}$  be a set of distinct points in a subset  $A \subseteq \mathbb{S}^d$ ,  $\eta > 0$ , and  $m \geq 0$  be an integer. The statement “The set  $\mathcal{C}$  admits a Marcinkiewicz–Zygmund type inequality on  $A$  for  $\Pi_m^d$  with constant  $\eta$ ” will mean the following. There exists a reduction of this data set, to be denoted again by  $\mathcal{C}$ , with its mesh norm with respect to  $A$  increased at most by a constant multiple, and a partition of  $A$ , each member of

which contains a unique  $\xi$  in the reduced set  $\mathcal{C}$ , and hence, can be denoted by  $R_\xi$ , with the following property:

$$\sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_d(\xi) \leq \eta \int_A |P(\mathbf{x})| d\mu_d(\mathbf{x}) \quad (3.12)$$

for all  $P \in \Pi_m^d$ .

In the context of the whole sphere, we proved the following (cf. [8, Corollary 3.1, Proposition 3.2]).

**Theorem 3.3.** *Let  $\mathcal{C}$  be a set of distinct points in  $\mathbb{S}^d$ ,  $n \geq 1$  be an integer, and  $\eta \in (0, 1)$ . There exists a constant  $c = c(d)$  with the following property. If*

$$\delta_d(\mathcal{C}, \mathbb{S}^d) \leq \frac{c\eta}{n} \quad (3.13)$$

*then the set  $\mathcal{C}$  admits a Marcinkiewicz–Zygmund type inequality on  $\mathbb{S}^d$  for  $\Pi_n^d$  with constant  $\eta$ ; i.e. for all  $P \in \Pi_n^d$ ,*

$$\sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_d(\mathbf{x}) \leq \eta \int_{\mathbb{S}^d} |P(\mathbf{x})| d\mu_d(\mathbf{x}). \quad (3.14)$$

Finally, we recall the Nikolskii inequalities, proved in [9, Proposition 2.1].

**Proposition 3.5.** *Let  $1 \leq p < r \leq \infty$ ,  $n \geq 1$  be an integer, and  $P \in \Pi_n^d$ . Then*

$$\|P\|_{\mathbb{S}^d, p} \leq c(d) \|P\|_{\mathbb{S}^d, r} \leq c(d) n^{d/p - d/r} \|P\|_{\mathbb{S}^d, p}, \quad (3.15)$$

*where the constant  $c(d)$  depends only on  $d$ .*

## 4. Proofs

### 4.1. The case $q = 1$

In this subsection, we will prove an analogue of Theorem 3.3 in the case of the circle; i.e., the case  $q = 1$ . This will help to clarify our ideas. It is also necessary for technical reasons to deal with this case separately.

**Theorem 4.1.** *Let  $J^\circ$  be an arc of the circle  $\mathbb{S}^1$ ,  $\mathcal{C}$  be a set of distinct points on  $J^\circ$ , and  $m \geq 1$  be an integer. Then there exists a constant  $c := c(q, m) > 0$  with the following property. If  $\eta > 0$  and  $\delta_1(\mathcal{C}, J^\circ) \leq c\eta |J^\circ|$ , then the set  $\mathcal{C}$  admits a Marcinkiewicz–Zygmund type inequality on  $J^\circ$  for  $\Pi_m^1$  with constant  $\eta$ ; i.e., for any polynomial  $P \in \Pi_m^1$ ,*

$$\sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_1(\mathbf{x}) \leq \eta \int_{J^\circ} |P(\mathbf{x})| d\mu_1(\mathbf{x}) \quad (4.1)$$

for a suitable reduction in  $\mathcal{C}$  and partition  $\{R_\xi\}$  of  $J^\circ$ . Equivalently, using the notation  $\xi = \exp(i\theta_\xi)$ ,  $J_\xi = \{\theta : \exp(i\theta) \in R_\xi\}$ ,  $J = \bigcup J_\xi$ , for any  $T \in \mathbb{H}_m$ ,

$$\sum_{\xi \in \mathcal{C}} \int_{J_\xi} |T(\theta) - T(\theta_\xi)| d\theta \leq \eta \int_J |T(\theta)| d\theta. \quad (4.2)$$

**Proof.** In this proof only, let  $\delta := \delta_1(\mathcal{C}, J^\circ)$ . We divide  $J^\circ$  into arcs of lengths between  $3\delta$  and  $6\delta$ , giving the partition  $\mathcal{P}$  of  $J^\circ$ . Then each arc contains at least one point of  $\mathcal{C}$  in its interior. In each arc  $R$ , we keep only one such point. We may then label the arc containing  $\xi$  by  $R_\xi$ . This yields the reduced data set as stated in the theorem, which will again be denoted by  $\mathcal{C}$ .

Now, let  $T \in \mathbb{H}_m$ . We have

$$\begin{aligned} \sum_{\xi \in \mathcal{C}} \int_{J_\xi} |T(\theta) - T(\theta_\xi)| d\theta &\leq \sum_{\xi \in \mathcal{C}} \int_{J_\xi} \int_{J_\xi} |T'(t)| dt d\theta \\ &\leq 6\delta \sum_{\xi \in \mathcal{C}} \int_{J_\xi} |T'(t)| dt = 6\delta \int_J |T'(t)| dt. \end{aligned}$$

Using Proposition 3.1, we conclude that

$$\sum_{\xi \in \mathcal{C}} \int_{J_\xi} |T(\theta) - T(\theta_\xi)| d\theta \leq \frac{cm^2\delta}{|J|} \int_J |T(\theta)| d\theta.$$

Therefore, we obtain (4.2) if  $\delta \leq \eta m^{-2} |J| = c\eta m^{-2} |J^\circ|$ .  $\square$

#### 4.2. The case $q \geq 2$

In this subsection, we obtain an analogue of Theorem 3.3 in the case of caps and bands on the sphere  $\mathbb{S}^q$  when  $q \geq 2$ . Without loss of generality, we may assume that the cap in question is centered at the north pole, and write  $\mathbb{S}_\alpha^q := \mathbb{S}_\alpha^q((0, \dots, 0, 1))$ .

Although our results are independent of the coordinate system used, we find it convenient to assume the standard parameterization of  $\mathbb{S}^q$  embedded in  $\mathbb{R}^{q+1}$  in terms of the angles  $\theta_1, \dots, \theta_q$ , where  $-\pi \leq \theta_1 \leq \pi$  and  $0 \leq \theta_k \leq \pi$  for  $k = 2, \dots, q$ . If  $\mathbf{x} \in \mathbb{S}^q$ , then the  $k$ th component of  $\mathbf{x}$  is given by

$$x_k = \begin{cases} \prod_{j=1}^q \sin \theta_j, & k = 1, \\ \cos \theta_{k-1} \prod_{j=k}^q \sin \theta_j, & 2 \leq k \leq q, \\ \cos \theta_q, & k = q + 1. \end{cases} \quad (4.3)$$

The measure  $\mu_q$  on  $\mathbb{S}^q$  can be expressed in these coordinates as

$$d\mu_q(\mathbf{x}) = \prod_{k=1}^q \sin^{k-1}(\theta_k) d\theta_k. \quad (4.4)$$

Note that

$$d\mu_q = \sin^{q-1}(\theta_q) d\theta_q d\mu_{q-1}. \quad (4.5)$$

We observe that any  $\mathbf{x} \in \mathbb{S}^q$  can be written in the form  $\sin \theta_q(\mathbf{x}', 0) + \cos \theta_q \mathbf{e}_{q+1}$ , where  $\mathbf{x}' \in \mathbb{S}^{q-1}$  and  $\mathbf{e}_{q+1}$  is the unit vector  $(0, \dots, 0, 1)$ . Accordingly, for any  $\mathbf{y} \in \mathbb{S}^{q-1}$  and  $\phi \in [0, \pi]$ , we write

$$[\mathbf{y}, \phi] := \sin \phi(\mathbf{y}, 0) + \cos \phi \mathbf{e}_{q+1} \in \mathbb{S}^q,$$

and for  $\mathbf{x} \in \mathbb{S}^q$ ,  $\mathbf{x} = [\mathbf{x}', \theta_q(\mathbf{x})]$ . We define  $\mathbf{e}'_{q+1} := \mathbf{e}_q \in \mathbb{S}^{q-1}$  and  $(-\mathbf{e}_{q+1})' := -\mathbf{e}_q \in \mathbb{S}^{q-1}$ .

In order to prove an analogue of Theorem 3.3, we wish to think of  $\mathbb{S}^q_\alpha$  as a cross product of  $\mathbb{S}^{q-1}$  with an arc. We would then like to use Theorem 3.3 for the spheres, and Theorem 4.1 for the arc, and combine them in a tensor product way. There are many technical problems, arising from the fact that the radii of different spheres at different levels are different, and hence, the definition of mesh norm on each sphere uses a different geodesic distance. To address this problem, we divide  $\mathbb{S}^q_\alpha$  into two parts. A small cap where only one point of  $\mathcal{C}$  is retained, and a band consisting of the remainder of  $\mathbb{S}^q_\alpha$ . This band will be further decomposed into thinner bands, each of which can be essentially thought of as a cross product of a small arc and a lower dimensional sphere (namely, the center of this subband), and will have a sufficiently dense subset of  $\mathcal{C}$  in its interior. We will make a careful reduction of this subset, projecting it on the center sphere of the subband, and lifting it up again. Theorem 3.3 will be used for the center sphere, and Theorem 4.1 for the arc. Along with Theorem 3.2 and our careful selection of the subset of  $\mathcal{C}$ , this approach will lead to the following analogue of Theorem 4.1 for the cap.

**Theorem 4.2.** *Let  $q \geq 2$ ,  $0 < \alpha < \pi$ ,  $\mathcal{C}$  be a set of distinct points in  $\mathbb{S}^q_\alpha$ , and  $m \geq 1$  be an integer. Then there exists a constant  $c := c(q, m) > 0$  with the following property. If  $\eta > 0$  and*

$$\delta(\mathcal{C}, \mathbb{S}^q_\alpha) \leq c\eta^{q/(q-1)}\alpha, \quad (4.6)$$

*then the set  $\mathcal{C}$  admits a Marcinkiewicz–Zygmund type inequality on  $\mathbb{S}^q_\alpha$  for  $\Pi_m^q$  with constant  $\eta$ ; i.e., for any polynomial  $P \in \Pi_m^q$ ,*

$$\sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_q(\mathbf{x}) \leq \eta \int_{\mathbb{S}^q_\alpha} |P(\mathbf{x})| d\mu_q(\mathbf{x}), \quad (4.7)$$

*for a suitable reduction in  $\mathcal{C}$  and partition  $\{R_\xi\}$  of  $\mathbb{S}^q_\alpha$ .*

In order to prove this theorem, we first have to prove its analogue for a spherical band. For  $0 \leq \beta < \gamma \leq \pi$ , the band  $\mathbb{S}^q_{\beta, \gamma}$  is defined by

$$\mathbb{S}^q_{\beta, \gamma} := \{\mathbf{x} \in \mathbb{S}^q : \beta \leq \theta_q(\mathbf{x}) \leq \gamma\}. \quad (4.8)$$

The cap  $\mathbb{S}^q_{0, \gamma}$  is denoted by  $\mathbb{S}^q_\gamma$ , and of course,  $\mathbb{S}^q_{0, \pi} = \mathbb{S}^q$ . The minimum radius of  $\mathbb{S}^q_{\beta, \gamma}$  is given by  $\rho_{\beta, \gamma} := \min(\sin \beta, \sin \gamma)$ .

**Theorem 4.3.** Let  $0 < \beta < \gamma < \pi$ ,  $\mathcal{C}$  be a set of distinct points in  $\mathbb{S}_{\beta,\gamma}^q$ , and  $m \geq 1$  be an integer. Then there exists a constant  $c := c(q, m) > 0$  with the following property. If  $\eta > 0$  and

$$\delta(\mathcal{C}, \mathbb{S}_{\beta,\gamma}^q) \leq c\eta \min(\rho_{\beta,\gamma}, \gamma - \beta), \quad (4.9)$$

then the set  $\mathcal{C}$  admits a Marcinkiewicz–Zygmund type inequality on  $\mathbb{S}_{\beta,\gamma}^q$  for  $\Pi_m^q$  with constant  $\eta$ ; i.e., for any polynomial  $P \in \Pi_m^q$ ,

$$\sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_q(\mathbf{x}) \leq \eta \int_{\mathbb{S}_{\beta,\gamma}^q} |P(\mathbf{x})| d\mu_q(\mathbf{x}), \quad (4.10)$$

for a suitable reduction in  $\mathcal{C}$  and partition  $\{R_\xi\}$  of  $\mathbb{S}_{\beta,\gamma}^q$ .

**Proof.** In this proof only, we adopt the following notations. Let  $\rho = \rho_{\beta,\gamma}$ ,

$$\delta := \delta_q(\mathcal{C}, \mathbb{S}_{\beta,\gamma}^q), \quad n := \left\lfloor \frac{\gamma - \beta}{3\delta} \right\rfloor,$$

and for  $k = 1, \dots, n$ ,

$$I_k := \left\{ \phi : \beta + \frac{k-1}{n}(\gamma - \beta) \leq \phi \leq \beta + \frac{k}{n}(\gamma - \beta) \right\}, \quad D_k := \{\mathbf{x} \in \mathbb{S}^q : \theta_q(\mathbf{x}) \in I_k\}.$$

Let  $1 \leq k \leq n$ ,  $\phi_k$  be the center of  $I_k$ , and  $\mathcal{C}_k$  be the collection of points in  $\mathcal{C}$  that lie in the interior of  $D_k$ . If  $\mathbf{y} \in \mathbb{S}^{q-1}$ , then by definition of the mesh norm, there exists  $\xi \in \mathcal{C}$  such that

$$\text{dist}_q([\mathbf{y}, \phi_k], \xi) \leq \delta \leq (\gamma - \beta)/(3n).$$

Necessarily,  $\xi$  is in the interior of  $D_k$ , and in particular, in  $\mathcal{C}_k$ . If there are more than one  $\xi \in \mathcal{C}_k$  which differ only in their  $\theta_q$  component, we keep only one of these, and denote the reduced set by  $\mathcal{C}_k$  again. Let  $\mathcal{C}'_k := \{\xi' \in \mathbb{S}^{q-1} : \xi \in \mathcal{C}_k\}$ . We observe that for every  $\xi' \in \mathcal{C}'_k$ , there exists a unique  $\xi \in \mathcal{C}_k$ . Next, we obtain a further reduction of  $\mathcal{C}'_k$  (and the corresponding reduction in  $\mathcal{C}_k$ ), and estimate the mesh norms of the reduced sets.

It is easy to check that for any  $\mathbf{y} \in \mathbb{S}^{q-1}$ , there exists  $\xi' \in \mathcal{C}'_k$  such that

$$\begin{aligned} \text{dist}_{q-1}(\mathbf{y}, \xi') &\leq c \frac{\text{dist}_q([\mathbf{y}, \phi_k], [\xi', \phi_k])}{\sin \phi_k} \\ &\leq \frac{c}{\sin \phi_k} (\text{dist}_q([\mathbf{y}, \phi_k], \xi) + \text{dist}_q(\xi, [\xi', \phi_k])) \leq \frac{c\delta}{\sin \phi_k}. \end{aligned}$$

Therefore,

$$\delta_{q-1}(\mathcal{C}'_k, \mathbb{S}^{q-1}) \leq \frac{c\delta}{\sin \phi_k} \leq \frac{c\delta}{\rho} \leq c\eta. \quad (4.11)$$

Theorem 3.3 yields a partition  $\mathcal{R}'_k$  of  $\mathbb{S}^{q-1}$  such that each member of this partition contains exactly one  $\xi' \in \mathcal{C}'_k$ . This member may be denoted by  $R_{\xi'}$ . This involves a further reduction of  $\mathcal{C}'_k$ , increasing its mesh norm with respect to  $\mathbb{S}^{q-1}$  only by a

constant multiple. We denote this reduced set again by  $\mathcal{C}'_k$ . The corresponding subset of  $\mathcal{C}_k$  will also be denoted again by  $\mathcal{C}_k$ . If  $\mathbf{x} \in D_k$ , then

$$\text{dist}_q(\mathbf{x}, [\mathbf{x}', \phi_k]) \leq \frac{\gamma - \beta}{2n} \leq c\delta. \quad (4.12)$$

Now, there exists  $\xi' \in \mathcal{C}'_k$  (after reduction), such that

$$\text{dist}_q([\mathbf{x}', \phi_k], [\xi', \phi_k]) \leq c\delta_{q-1}(\mathcal{C}'_k, \mathbb{S}^{q-1}) \sin \phi_k \leq c\delta,$$

where the last inequality follows from (4.11). There is a unique  $\xi \in \mathcal{C}_k$  with this  $\xi'$ , and using (4.12) again with  $\xi$  in place of  $\mathbf{x}$ , we conclude that  $\text{dist}_q(\mathbf{x}, \xi) \leq c\delta$ . Thus, the mesh norm of the reduced set  $\mathcal{C}_k$  satisfies  $\delta_q(\mathcal{C}_k, D_k) \leq c\delta$ .

Finally, we write  $R_\xi := \{[\mathbf{x}', \phi] : \mathbf{x}' \in \mathcal{R}_{\xi'}, \phi \in I_k\}$ ,  $\xi \in \mathcal{C}_k$ , and observe that

$$\mu_q(R_\xi) = v_k \mu_{q-1}(R_{\xi'}), \quad v_k := \int_{I_k} \sin^{q-1} \phi \, d\phi. \quad (4.13)$$

Moreover,  $\{R_\xi\}$  is a partition of  $D_k$ , each member of which contains exactly one element of  $\mathcal{C}_k$ . If  $\theta, \phi \in I_k$ , then

$$|\log \sin \theta - \log \sin \phi| \leq \frac{|\theta - \phi|}{\rho} \leq \frac{\gamma - \beta}{n\rho} \leq c\delta/\rho.$$

Therefore, if  $\delta/\rho \leq c_1$ , then  $\sin \theta \sim \sin \phi$  for all  $\theta, \phi \in I_k$ . Consequently, for any integrable function  $f$  on  $I_k$ ,

$$v_k \int_{I_k} |f(t)| \, dt \leq c|I_k| \int_{I_k} |f(t)| \sin^{q-1} t \, dt \leq c\delta \int_{I_k} |f(t)| \sin^{q-1} t \, dt. \quad (4.14)$$

Having obtained a partition of  $D_k$ , (and thus, also of the whole band  $\mathbb{S}_{\beta, \gamma}^q$ ), we now turn to polynomial inequalities.

Theorem 3.3 and (4.11) imply that if  $\delta/\rho \leq c\eta$ ,

$$\sum_{\xi' \in \mathcal{C}'_k} \int_{R_{\xi'}} |\mathcal{Q}(\mathbf{y}) - \mathcal{Q}(\xi')| \, d\mu_{q-1}(\mathbf{y}) \leq \eta \int_{\mathbb{S}^{q-1}} |\mathcal{Q}(\mathbf{y})| \, d\mu_{q-1}(\mathbf{y}) \quad (4.15)$$

for any  $\mathcal{Q} \in \Pi_m^{q-1}$ .

Now, let  $P \in \Pi_m^q$ . We have

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| \, d\mu_q(\mathbf{x}) \\ & \leq \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P(\mathbf{x}) - P([\mathbf{x}', \phi_k])| \, d\mu_q(\mathbf{x}) \\ & \quad + \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\mathbf{x}', \phi_k]) - P([\xi', \phi_k])| \, d\mu_q(\mathbf{x}) \\ & \quad + \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\xi', \phi_k]) - P(\xi)| \, d\mu_q(\mathbf{x}). \end{aligned} \quad (4.16)$$

It is convenient to estimate the middle term first. Using (4.15) with  $Q(\mathbf{y}) := P([\mathbf{y}, \phi_k])$  and (4.13), we obtain that

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\mathbf{x}', \phi_k]) - P([\xi', \phi_k])| d\mu_q(\mathbf{x}) \\ &= v_k \sum_{\xi' \in \mathcal{C}'_k} \int_{R_{\xi'}} |P([\mathbf{x}', \phi_k]) - P([\xi', \phi_k])| d\mu_{q-1}(\mathbf{x}') \\ &\leq \eta v_k \int_{\mathbb{S}^{q-1}} |P([\mathbf{x}', \phi_k])| d\mu_{q-1}(\mathbf{x}'). \end{aligned} \quad (4.17)$$

Since  $P([\mathbf{x}', t])$  is a trigonometric polynomial in  $t$  for every  $\mathbf{x}'$ , we may use (3.3) to obtain

$$v_k |P([\mathbf{x}', \phi_k])| \leq \frac{cv_k}{|I_k|} \int_{I_k} |P([\mathbf{x}', t])| dt.$$

Using (4.14), we get

$$v_k |P([\mathbf{x}', \phi_k])| \leq c \int_{I_k} |P([\mathbf{x}', t])| \sin^{q-1} t dt.$$

Along with (4.17), this leads to

$$\sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\mathbf{x}', \phi_k]) - P([\xi', \phi_k])| d\mu_q(\mathbf{x}) \leq c\eta \int_{D_k} |P(\mathbf{x})| d\mu_q(\mathbf{x}). \quad (4.18)$$

Next, we estimate the last term of the right-hand side of (4.16). Using (4.13), we get

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\xi', \phi_k]) - P(\xi)| d\mu_q(\mathbf{x}) \\ &\leq \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} \int_{I_k} \left| \frac{\partial}{\partial t} P([\xi', t]) \right| dt d\mu_q(\mathbf{x}) \\ &= v_k \int_{I_k} \sum_{\xi' \in \mathcal{C}'_k} \mu_{q-1}(R'_\xi) \left| \frac{\partial}{\partial t} P([\xi', t]) \right| dt \end{aligned} \quad (4.19)$$

Now, using (4.15) applied with  $\frac{\partial}{\partial t} P([\cdot, t])$  in place of  $Q$ , we deduce that

$$\sum_{\xi' \in \mathcal{C}'_k} \mu_{q-1}(R_{\xi'}) \left| \frac{\partial}{\partial t} P([\xi', t]) \right| \leq c \int_{\mathbb{S}^{q-1}} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_{q-1}(\mathbf{x}').$$

Using (4.14), we get that

$$v_k \int_{I_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| dt \leq c\delta \int_{I_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| \sin^{q-1} t dt. \quad (4.20)$$

Substituting from the last two estimates into (4.19), we conclude that

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} \int_{I_k} \left| \frac{\partial}{\partial t} P([\xi', t]) \right| dt d\mu_q(\mathbf{x}) \\ & \leq c\delta \int_{\mathbb{S}^{q-1}} \int_{I_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_{q-1}(\mathbf{x}') \sin^{q-1} t dt \\ & = c\delta \int_{D_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}), \end{aligned}$$

and hence, that

$$\sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P([\xi', \phi_k]) - P(\xi)| d\mu_q(\mathbf{x}) \leq c\delta \int_{D_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}). \quad (4.21)$$

Finally, using (4.20), we get

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P(\mathbf{x}) - P([\mathbf{x}', \phi_k])| d\mu_q(\mathbf{x}) \\ & \leq \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} \int_{I_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| dt d\mu_q(\mathbf{x}) \\ & = v_k \int_{\mathbb{S}^{q-1}} \int_{I_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| dt d\mu_{q-1}(\mathbf{x}') \\ & \leq c\delta \int_{D_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}). \end{aligned} \quad (4.22)$$

Substituting from the estimates (4.22), (4.21), and (4.18) into (4.16), we arrive at

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_q(\mathbf{x}) \\ & \leq c\delta \int_{D_k} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}) + c\eta \int_{D_k} |P(\mathbf{x})| d\mu_q(\mathbf{x}). \end{aligned} \quad (4.23)$$

Adding estimates (4.23) for  $k = 1, \dots, n$ , we obtain

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_q(\mathbf{x}) \\ & \leq \sum_{k=1}^n \sum_{\xi \in \mathcal{C}_k} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| d\mu_q(\mathbf{x}) \\ & \leq c\eta \int_{\mathbb{S}_{\beta, \gamma}^q} |P(\mathbf{x})| d\mu_q(\mathbf{x}) + c\delta \int_{\mathbb{S}_{\beta, \gamma}^q} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}). \end{aligned} \quad (4.24)$$



Now, using the inequality (3.6) with  $P([\mathbf{x}', t])$ , which is a trigonometric polynomial of degree at most  $m$  in  $t$ , and using condition (4.9), we deduce that

$$\begin{aligned} \delta \int_{\beta}^{\gamma} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| \sin^{q-1} t \, dt &\leq \frac{c\delta}{\gamma - \beta} \int_{\beta}^{\gamma} |P([\mathbf{x}', t])| \sin^{q-1} t \, dt \\ &\leq c\eta \int_{\beta}^{\gamma} |P([\mathbf{x}', t])| \sin^{q-1} t \, dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &\delta \int_{\mathbb{S}_{\beta, \gamma}^q} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| d\mu_q(\mathbf{x}) \\ &= \delta \int_{\mathbb{S}^{q-1}} \int_{\beta}^{\gamma} \left| \frac{\partial}{\partial t} P([\mathbf{x}', t]) \right| \sin^{q-1} t \, dt \, d\mu_{q-1}(\mathbf{x}') \\ &\leq c\eta \int_{\mathbb{S}^{q-1}} \int_{\beta}^{\gamma} |P([\mathbf{x}', t])| \sin^{q-1} t \, dt \, d\mu_{q-1}(\mathbf{x}') \\ &= c\eta \int_{\mathbb{S}_{\beta, \gamma}^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}). \end{aligned}$$

Together with (4.24), this leads to (4.10).  $\square$

We are now able to complete the proof of Theorem 4.2.

**Proof of Theorem 4.2.** In this proof only, we write  $\delta := \delta(\mathcal{C}, \mathbb{S}_{\alpha}^q)$ . Let  $\tau := (2\delta/\alpha)^{1/q}$ , and we assume that  $\tau < 1/2$ . Then  $\tau > \tau^q$ ; i.e.,  $\tau\alpha > 2\delta$ , and the cap  $\mathbb{S}_{\tau\alpha}^q$  contains at least one point of  $\mathcal{C}$ , say  $\xi_0$ . We keep only one such point in the cap, and hence, will denote the cap also by  $R_{\xi_0}$ . In this proof only, let  $\mathbf{n} := (0, \dots, 0, 1)$ . Let  $P \in \Pi_m^q$ . Then using (3.4) and (3.6), we see that for any  $\mathbf{y} \in \mathbb{S}_{\tau\alpha}^q$ ,

$$\begin{aligned} |P(\mathbf{y}) - P(\mathbf{n})| &\leq \int_0^{\alpha} \left| \frac{\partial}{\partial t} P([\mathbf{y}', t]) \right| \, dt \\ &\leq c\alpha^{-q+1} \int_0^{\alpha} \left| \frac{\partial}{\partial t} P([\mathbf{y}', t]) \right| \sin^{q-1} t \, dt \\ &\leq c\alpha^{-q} \int_0^{\alpha} |P([\mathbf{y}', t])| \sin^{q-1} t \, dt. \end{aligned} \tag{4.25}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{S}_{\tau\alpha}^q} |P(\mathbf{y}) - P(\mathbf{n})| \, d\mu_q(\mathbf{y}) \\ &\leq c\alpha^{-q} \int_0^{\tau\alpha} \int_{\mathbb{S}^{q-1}} \int_0^{\alpha} |P([\mathbf{y}', t])| \sin^{q-1} t \, dt \, d\mu_{q-1}(\mathbf{y}') \sin^{q-1} \theta_q(\mathbf{y}) \, d\theta_q(\mathbf{y}) \\ &\leq c\tau^q \int_{\mathbb{S}_{\alpha}^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}). \end{aligned} \tag{4.26}$$

Next, using (4.25) and the Nikolskii inequality (3.3), we see that

$$\begin{aligned} |P(\xi_0) - P(\mathbf{n})| &\leq c\alpha^{-q} \int_0^\alpha |P([\xi'_0, t])| \sin^{q-1} t \, dt \\ &\leq c\alpha^{-q} \int_0^\alpha \int_{\mathbb{S}^{q-1}} |P([\mathbf{y}, t])| \sin^{q-1} t \, d\mu_{q-1}(\mathbf{y}) \, dt \\ &= c\alpha^{-q} \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}). \end{aligned}$$

Along with (4.26) and condition (4.6), this implies that

$$\begin{aligned} \int_{\mathbb{S}_{\tau\alpha}^q} |P(\mathbf{y}) - P(\xi_0)| \, d\mu_q(\mathbf{y}) &\leq c\tau^q \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}) \\ &\leq c\eta \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}). \end{aligned} \quad (4.27)$$

Next, we apply Theorem 4.3 with  $\mathcal{C} \setminus \{\xi_0\}$  in place of  $\mathcal{C}$  and  $\mathbb{S}_{\tau\alpha, x}^q$ . It is not difficult to verify using triangle inequality that

$$\delta_q(\mathcal{C} \setminus \{\xi_0\}, \mathbb{S}_{\tau\alpha, x}^q) \leq c\delta.$$

Therefore, condition (4.6) ensures that (4.9) is satisfied with  $\mathcal{C} \setminus \{\xi_0\}$  in place of  $\mathcal{C}$ . Accordingly, we get a partition  $\{R_\xi : \xi \in \mathcal{C} \setminus \{\xi_0\}\}$  of  $\mathbb{S}_{\tau\alpha, x}^q$  such that

$$\sum_{\xi \in \mathcal{C} \setminus \{\xi_0\}} \int_{R_\xi} |P(\xi) - P(\xi)| \, d\mu_q(\mathbf{x}) \leq \eta \int_{\mathbb{S}_{\tau\alpha, x}^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}) \leq c\eta \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}).$$

Recalling that  $\mathbb{S}_{\tau\alpha}^q =: R_{\xi_0}$ , and adding (4.27) to the above estimate, we arrive at the desired partition of  $\mathbb{S}_x^q$  for which (4.7) holds.  $\square$

### 4.3. Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1. We use Theorem 3.1 with  $\Pi_m^q$  in place of  $X$ , take  $Z$  to be the set of point evaluation functionals at points of  $\mathcal{C}$ , and  $x^*$  to be the functional associating each polynomial with its integral over the cap. We observe that the polynomial identically equal to 1 serves in place of  $x_0$  in Theorem 3.1. It remains to prove that  $Z$  is a norming set for  $X$  and  $x^*$  is positive with respect to  $Z$ .

In (4.7) ((4.1) in the case  $q = 1$ ), we choose  $\eta = 1/3$ . Then

$$\begin{aligned} &\left| \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}) - \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) |P(\xi)| \right| \\ &\leq \sum_{\xi \in \mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)| \, d\mu_q(\mathbf{x}) \\ &\leq (1/3) \int_{\mathbb{S}_x^q} |P(\mathbf{x})| \, d\mu_q(\mathbf{x}). \end{aligned} \quad (4.28)$$

Hence,

$$\int_{\mathbb{S}_x^q} |P(\mathbf{x})| d\mu_q(\mathbf{x}) \leq \frac{3}{2} \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) |P(\xi)| \leq 2 \int_{\mathbb{S}_x^q} |P(\mathbf{x})| d\mu_q(\mathbf{x}). \quad (4.29)$$

Therefore, if each  $P(\xi) = 0$ , necessarily,  $P = 0$ ; i.e.,  $Z$  is a norming set. Next, let each  $P(\xi) \geq 0$ ,  $\xi \in \mathcal{C}$ . Then (4.28) and (4.29) together imply that

$$\left| \int_{\mathbb{S}_x^q} P(\mathbf{x}) d\mu_q(\mathbf{x}) - \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) P(\xi) \right| \leq (1/3) \int_{\mathbb{S}_x^q} |P(\mathbf{x})| d\mu_q(\mathbf{x}) \leq \frac{1}{2} \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) P(\xi).$$

Therefore,

$$\int_{\mathbb{S}_x^q} P(\mathbf{x}) d\mu_q(\mathbf{x}) \geq \frac{1}{2} \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) P(\xi) \geq 0.$$

Thus,  $x^*$  is positive with respect to  $Z$ .

This completes the proof of Theorem 2.1.

The same proof as above, using (4.10) in place of (4.7) implies the following theorem.

**Theorem 4.4.** *Let  $q \geq 1$  and  $m \geq 1$  be integers,  $0 < \beta < \gamma < \pi$ . There exists a positive constant  $c_1$ , depending only on  $q$  and  $m$  (but independent of  $\beta$  and  $\gamma$ ) with the following property. If  $\mathcal{C} \subseteq \mathbb{S}_{\beta, \gamma}^q$ , and*

$$\delta_q(\mathcal{C}, \mathbb{S}_{\beta, \gamma}^q) \leq c_1 \min(\rho_{\beta, \gamma}, \gamma - \beta), \quad (4.30)$$

*then there exist nonnegative weights  $w_\xi$  such that*

$$\sum_{\xi \in \mathcal{C}} w_\xi P(\xi) = \int_{\mathbb{S}_{\beta, \gamma}^q} P(\mathbf{x}) d\mu_q(\mathbf{x}), \quad P \in \Pi_m^q. \quad (4.31)$$

*Moreover,  $|\{\xi \in \mathcal{C} : w_\xi \neq 0\}| \leq c$ , and hence,*

$$\max_{\xi \in \mathcal{C}} w_\xi \sim \sum_{\xi \in \mathcal{C}} w_\xi \sim \mu_q(\mathbb{S}_{\beta, \gamma}^q). \quad (4.32)$$

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